D-branes and Deep Learning

Theoretical and Computational Aspects in String Theory

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Action Principle and Conformal Symmetry

Polyakov's Action

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Symmetries:

- **Poincaré transf.** $X'^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu$
- **2D diff.** $\gamma'_{\alpha\beta} = \left(J^{-1}\right)_{\alpha\beta}{}^{\lambda\rho} \gamma_{\lambda\rho}$
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Conformal symmetry:

- **vanishing** stress-energy tensor: $\mathcal{T}_{\alpha\beta} = 0$
- **traceless** stress-energy tensor: $\operatorname{tr}\mathcal{T} = 0$
- **conformal gauge** $\gamma_{\alpha\beta} = e^{\phi} \eta_{\alpha\beta}$

Action Principle and Conformal Symmetry

Superstrings in D dimensions:

$$\mathcal{T}(z) = -\frac{1}{\alpha'}\partial X(z)\cdot\partial X(z) - \frac{1}{2}\psi(z)\cdot\partial\psi(z) \quad \Rightarrow \quad c = \frac{3}{2}D$$

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$(\lambda, 0)$ / $(1 - \lambda, 0)$ Ghost System

Introduce anti-commuting $(b,\ c)$ and commuting $(\beta,\ \gamma)$ conformal fields:

$$S_{\text{ghost}}[b,\ c,\ \beta,\ \gamma] = \frac{1}{2\pi} \iint \mathrm{d}z\, \mathrm{d}\bar{z}\, \left(b(z)\, \bar{\partial} c(z) + \beta(z)\, \bar{\partial}\gamma(z) \right)$$

where $\lambda_b = 2$ and $\lambda_c = -1$, and $\lambda_\beta = \frac{3}{2}$ and $\lambda_\gamma = -\frac{1}{2}$.

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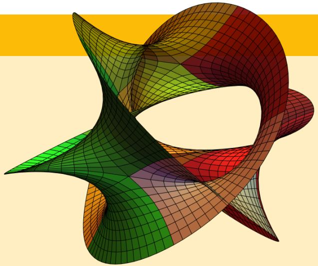
Consequence:

$$c_{\text{full}} = c + c_{\text{ghost}} = 0 \quad \Leftrightarrow \quad D = 10.$$

Extra Dimensions and Compactification

Compactification

$$\mathscr{M}^{1,9} = \mathscr{M}^{1,3} \otimes \mathscr{X}_6$$



- $\mathscr{X}_6$ is a **compact** manifold
- $N = 1$ **supersymmetry** is preserved in 4D
- algebra of $\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1)$ in arising **gauge group**

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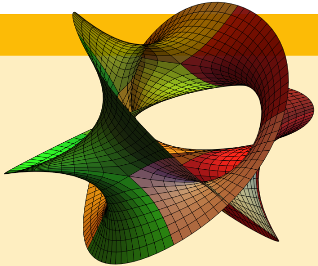

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Kähler manifolds (M, g) such that

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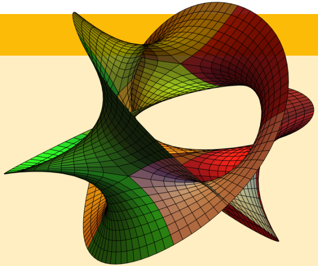

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Characterised by Hodge numbers

$$h^{r,s} = \dim_{\mathbb{C}} H_{\bar{\partial}}^{r,s}(M, \mathbb{C})$$

counting the no. of harmonic (r, s)-forms.

D-branes and Open Strings

Polyakov's action naturally introduces Neumann b.c.:

$$\partial_\sigma X(\tau, \sigma)\Big|_{\sigma=0}^{\sigma=\ell} = 0$$

satisfied by **open and closed** strings living in D dimensions s.t. $\Box X = 0$.

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T-duality

$$X(z, \bar{z}) = X(z) + \overline{X}(\bar{z}) \quad \xRightarrow{T} \quad X(z) - \overline{X}(\bar{z}) = Y(z, \bar{z}) = Y(z) + \overline{Y}(\bar{z})$$

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Resulting effect (repeated $p \leq D-1$ times) leads to Dirichlet b.c.:

$$\partial_\sigma X^i(\tau,\sigma)\Big|_{\sigma=0}^{\sigma=\ell} = 0 \quad \overset{T}{\Rightarrow} \quad \partial_\tau Y^i(\tau,\sigma)\Big|_{\sigma=0}^{\sigma=\ell} = 0 \quad \forall i = 1, 2, \dots, p$$

thus **open strings** can be **constrained** to $D(D-p-1)$-branes.

D-branes and Open Strings

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$$\mathcal{A}^{\mu} \quad \leftrightarrow \quad \alpha^{\mu}_{-1} |0\rangle \qquad \longrightarrow \qquad \begin{array}{lcll} \mathcal{A}^{A} & \leftrightarrow & \alpha^{A}_{-1} |0\rangle\,, & A = 0,\, 1,\, \dots,\, p \\ \mathcal{A}^{a} & \leftrightarrow & \alpha^{a}_{-1} |0\rangle\,, & a = 1,\, 2,\, \dots,\, D-p-1 \end{array}$$

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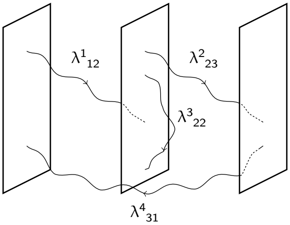

Introducing **Chan–Paton factors** $\lambda^r{}_{ij}$, when branes are **coincident**:

$$\bigoplus_{r=1}^{N} \mathrm{U}_r(1) \quad \longrightarrow \quad \mathrm{U}(N)$$

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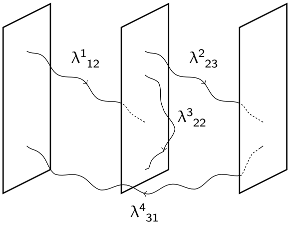

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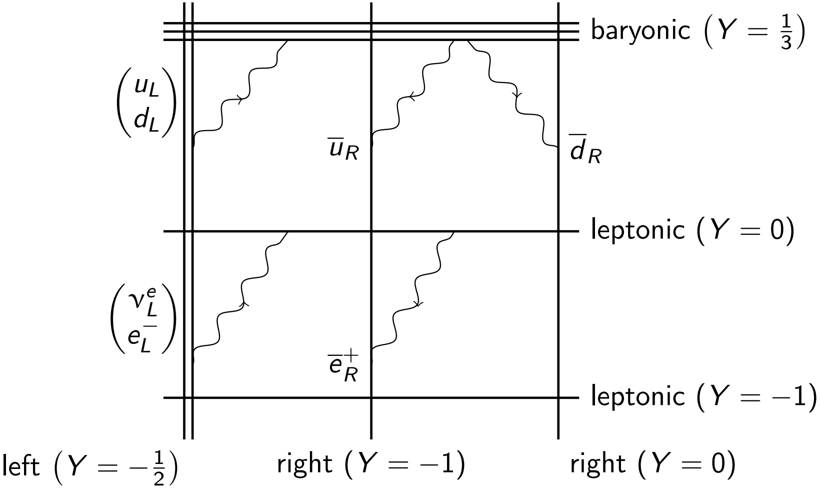
$\boxed{\text{Build gauge bosons, fermions and scalars.}}$

Standard Model-like Scenarios



Intersecting D-branes

Consider N intersecting $D6$-branes filling $\mathscr{M}^{1,3}$ and **embedded in** $\mathbb{R}^6$

Twist Fields Correlators

$$\left\langle \prod_{t=1}^{N_B} \sigma_{\mathrm{M}_{(t)}}\left(x_{(t)}\right) \right\rangle = \mathcal{N}\left(\left\{ x_{(t)},\, \mathrm{M}_{(t)} \right\}_{1 \leq t \leq N_B} \right) e^{-S_{E\,(\mathbf{cl})}\left(\left\{ x_{(t)},\, \mathrm{M}_{(t)} \right\}_{\mathbf{1} \leq t \leq N_B} \right)}$$

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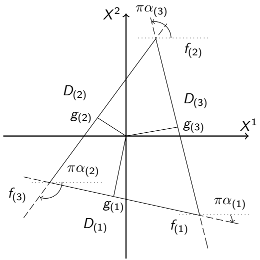

D-branes in **factorised** internal space:

- **embedded as lines** in $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$
- **relative rotations** are $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ elements
- $S_{E\,(\mathrm{cl})}\left(\left\{x_{(t)},\, \mathrm{M}_{(t)}\right\}_{1 \le t \le N_B}\right) \sim \mathrm{Area}\left(\left\{f_{(t)},\, \mathrm{R}_{(t)}\right\}_{1 \le t \le N_B}\right)$

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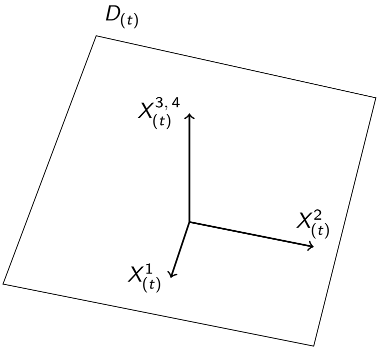

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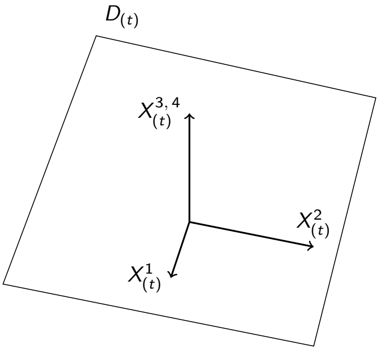

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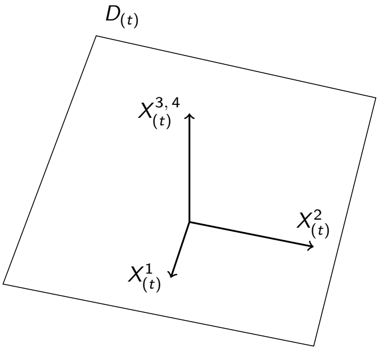

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that is

$$\left[R_{(t)}\right] = \left\{R_{(t)} \sim \mathcal{O}_{(t)} R_{(t)}\right\}$$

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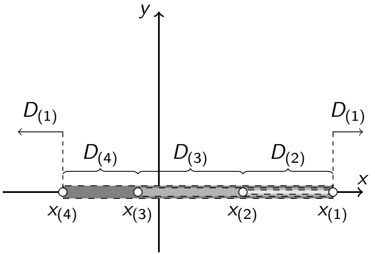
- consider $u = x + iy = e^{\tau_e + i\sigma}$ and $\overline{u} = u^*$
- let $x_{(t)} < x_{(t-1)}$ be the **worldsheet intersection points** on **real axis**
- $X^{1,2}_{(t)}$ are **Neumann**, $X^{3,4}_{(t)}$ are **Dirichlet**



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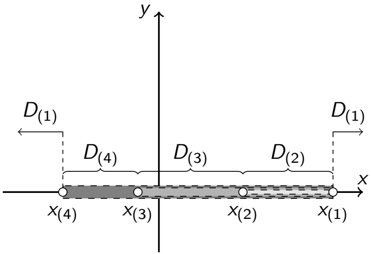

Branch Cuts and Discontinuities for $x \in D_{(t)}$

$$\begin{cases} \partial_u X(x + i0^+) & = U_{(t)} \cdot \partial_{\overline{u}} \overline{X}(x - i0^+) = \left[R^{-1}_{(t)} \cdot (\sigma_3 \otimes \mathbb{1}_2) \cdot R_{(t)} \right] \cdot \partial_{\overline{u}} \overline{X}(x - i0^+) \\ X(x_{(t)}, x_{(t)}) & = f_{(t)} \end{cases}$$

Doubling Trick and Spinor Representation

Doubling Trick

$$\partial_z \mathcal{X}(z) = \begin{cases} \partial_u X(u) & \text{if } z \in \mathscr{H}_{>}^{(\bar{t})} \\ U_{(\bar{t})}\, \partial_{\overline{u}} \overline{X}(\overline{u}) & \text{if } z \in \mathscr{H}_{<}^{(\bar{t})} \end{cases} \quad \Rightarrow \quad \begin{aligned} \partial_z \mathcal{X}(x_{(t)} + e^{2\pi i} \delta_+) &= \mathcal{U}_{(t,\,t+1)}\, \partial_z \mathcal{X}(x_{(t)} + \delta_+), \\ \partial_z \mathcal{X}(x_{(t)} + e^{2\pi i} \delta_-) &= \widetilde{\mathcal{U}}_{(t,\,t+1)}\, \partial_z \mathcal{X}(x_{(t)} + \delta_-), \end{aligned}$$

where $\mathscr{H}_{\gtrless}^{(t)} = \left\{ z \in \mathbb{C} \mid \operatorname{Im} z \gtrless 0 \text{ or } z \in D_{(t)} \right\}$ and $\delta_\pm = \eta \pm i0^+$.

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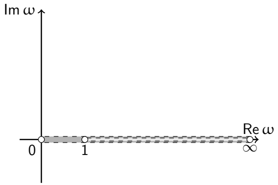
Use Pauli matrices $\tau = (i\, \mathbb{1}_2, \vec{\sigma})$:

$$\partial_z \mathcal{X}_{(s)}(z) = \partial_z \mathcal{X}^{I}(z)\, \tau_I \quad \Rightarrow \quad \partial_z \mathcal{X}(x_{(t)} + e^{2\pi i}\, \delta_{\pm}) = \overset{(\sim)}{\mathcal{L}}_{(t,\, t+1)}\, \partial_z \mathcal{X}(x_{(t)} + \delta_{\pm})\, \overset{(\sim)}{\mathcal{R}}_{(t,\, t+1)}$$

where

$$\overset{(\sim)}{\mathcal{L}}_{(t,\, t+1)} \in \mathrm{SU}(2)_L \quad \text{and} \quad \overset{(\sim)}{\mathcal{R}}_{(t,\, t+1)} \in \mathrm{SU}(2)_R$$

Hypergeometric Basis



Sum over all contributions:

$$\partial_z \mathcal{X}(z) = \sum_{l,\, r} c_{lr} \left(-\omega_z\right)^{A_{lr}} \left(1 - \omega_z\right)^{B_{lr}} B^{(L)}_{0,\, l}(\omega_z) \left(B^{(R)}_{0,\, r}(\omega_z)\right)^T$$

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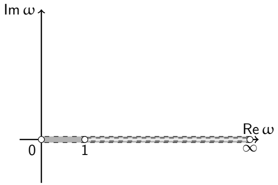

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Basis of Solutions

$$B_{0,\, n}(\omega_z) = \begin{pmatrix} 1 & 0 \\ 0 & K_n \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(c_n)}\, {}_2F_1\left(a_n,\, b_n;\, c_n;\, \omega_z\right) \\ \left(-\omega_z\right)^{1-c_n} \frac{1}{\Gamma(2-c_n)}\, {}_2F_1\left(a_n + 1 - c_n,\, b_n + 1 - c_n;\, 2 - c_n;\, \omega_z\right) \end{pmatrix}$$

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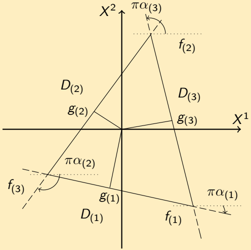
Physical Interpretation



$$
\begin{aligned}
S_{\mathbb{R}^4}\Big|_{\text{on-shell}} &= \frac{1}{2\pi\alpha'} \sum_{t=1}^{3} \left(\frac{1}{2} \left| g_{(t)}^{\perp} \right| \left| f_{(t-1)} - f_{(t)} \right| \right) \\
&= \mathrm{Area}\left(\left\{ f_{(t)} \right\} \right)
\end{aligned}
$$

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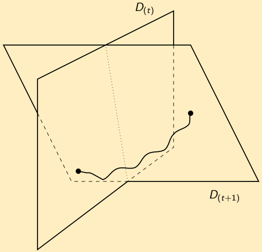
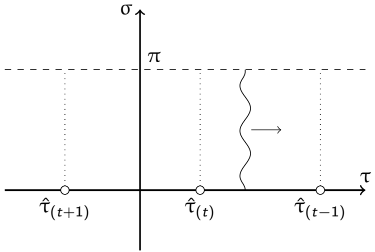
Operations sequence:

1. rotation matrix = monodromy matrix
2. contiguity relations $\Rightarrow$ independent hypergeometrics
3. finite action $\Rightarrow$ 2 solutions (no. of d.o.f. is correctly saturated)
4. boundary conditions $\Rightarrow$ fix free constants c_{lr}

Physical Interpretation



Fermions on the Strip



Action of Boundary Changing Operators

$$\begin{cases} \psi^i_-(\tau,0) & = \left(R_{(t)}\right)^I{}_J \psi^J_+(\tau,0) \quad \text{for } \tau \in \left(\hat{\tau}_{(t)},\, \hat{\tau}_{(t-1)}\right) \\ \psi^I_-(\tau,\pi) & = -\psi^I_+(\tau,\pi) \quad \text{for } \tau \in \mathbb{R} \end{cases}$$

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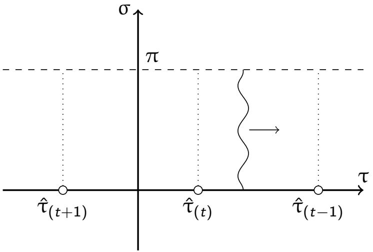

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Stress-energy Tensor

$$\mathcal{T}_{\pm\pm}(\xi_\pm) = -i\,\frac{T}{4}\,\psi^*_{\pm,\,I}(\xi_\pm)\,\overset{\leftrightarrow}{\partial}\psi^I_\pm(\xi_\pm) \quad \Rightarrow \quad \begin{cases} \dot{\mathrm{H}}(\tau) & = 0 \Leftrightarrow \tau \in \left(\tau_{(t)},\, \tau_{(t-1)}\right) \\ \dot{\mathrm{P}}(\tau) & \neq 0 \end{cases}$$

Conserved Product and Operators

Expand on a basis of solutions

$$\psi_{\pm}(\xi_{\pm}) = \sum_{n=-\infty}^{+\infty} b_n \, \psi_n(\xi_{\pm}) \qquad \Rightarrow \qquad \Psi(z) = \begin{cases} \psi_{E,+}(u) & \text{if } z \in \mathscr{H}_{>}^{(\bar{t})} \\ \psi_{E,-}(u) & \text{if } z \in \mathscr{H}_{<}^{(\bar{t})} \end{cases}$$

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$$\langle\!\langle {}^*\psi_n, \psi_m \rangle = 2\pi\mathcal{N} \oint \frac{\mathrm{d}z}{2\pi i} \, {}^*\Psi_n{}^* \, \Psi_m = \delta_{n,m} \qquad \Rightarrow \qquad \left\langle\!\!\left\langle {}^*\Psi_n{}^{(*)}, \Psi^{(*)} \right\rangle\right. = b_n^{(\dagger)}$$

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Derive the algebra of operators:

$$\left[b_n, b_m^{\dagger}\right]_{+} = \frac{2\mathcal{N}}{T} \, \langle\!\langle {}^{*}\Psi_n{}^{*}, \Psi_m^{*} \rangle$$

Twisted Complex Fermions

Consider the case $R_{(t)} = e^{i\pi\alpha_{(t)}} \in \mathrm{U}(1)$:

$$\Psi(x_{(t)} + e^{2\pi i}\delta) = e^{i\pi\epsilon_{(t)}}\,\Psi(x_{(t)} + \delta)$$

where

$$\epsilon_{(t)} = \alpha_{(t+1)} - \alpha_{(t)} + \theta\big(\alpha_{(t)} - \alpha_{(t+1)} - 1\big) - \theta\big(\alpha_{(t+1)} - \alpha_{(t)} - 1\big)$$

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Basis of Solutions

$$\Psi_n\big(z;\,\{x_{(t)}\}\big) = \mathcal{N}_\Psi\, z^{-n} \prod_{t=1}^{N} \left(1 - \frac{z}{x_{(t)}}\right)^{n_{(t)} + \frac{\epsilon_{(t)}}{2}}$$

$$^{*}\Psi_n\big(z;\,\{x_{(t)}\}\big) = \frac{1}{2\pi\mathcal{N}\mathcal{N}_\Psi}\, z^{n-1} \prod_{t=1}^{N} \left(1 - \frac{z}{x_{(t)}}\right)^{-\widetilde{n}_{(t)} + \frac{\epsilon_{(t)}}{2}}$$

Vacua

Define the **vacuum** with respect to b_n:

$$b_n \left| \left\{ x_{(t)} \right\} \right\rangle = 0 \quad \text{for} \quad n \geq 1$$
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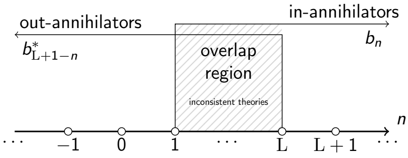

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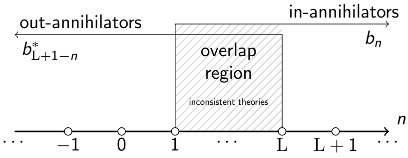

Stress-energy Tensor and CFT Approach

Compute the OPEs leading to the stress-energy tensor:

$$\mathcal{T}(z) = \frac{\pi T}{2} \mathcal{N}_\Psi^2 \sum_{n,\, m=-\infty}^{+\infty} :\, b_n\, b_m^* :\; z^{-n-m} \left[\frac{m-n}{2} + 2 \sum_{t=1}^{N} \frac{n_{(t)} + \frac{\epsilon_{(t)}}{2}}{z - x_{(t)}} \right] + \frac{1}{2} \left(\sum_{t=1}^{N} \frac{n_{(t)} + \frac{\epsilon_{(t)}}{2}}{z - x_{(t)}} \right)^2$$

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Invariant Vacuum and Spin Fields

$$\left| \left\{ x_{(t)} \right\} \right\rangle = \mathcal{N}\left(\left\{ x_{(t)} \right\} \right) \mathrm{R} \left[\prod_{t=1}^{M} S_{(t)}(x_{(t)}) \right] \left| 0 \right\rangle_{\mathrm{SL}_2(\mathbb{R})}$$

Spin Fields Amplitudes

Equivalence with Bosonization

$$\partial_{x_{(t)}} \left\langle \left\{ x_{(t)} \right\} \middle| \left\{ x_{(t)} \right\} \right\rangle = \oint\limits_{x_{(t)}} \frac{\mathrm{d}z}{2\pi i} \frac{\left\langle \left\{ x_{(t)} \right\} \middle| \mathcal{T}(z) \middle| \left\{ x_{(t)} \right\} \right\rangle}{\left\langle \left\{ x_{(t)} \right\} \middle| \left\{ x_{(t)} \right\} \right\rangle}$$

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- alternative framework for amplitudes (extension to (non) Abelian twist/spin fields?)

A Few Words on a Theory of Everything

string theory = theory of everything = nuclear forces + gravity

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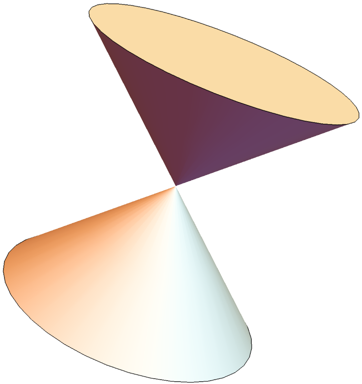

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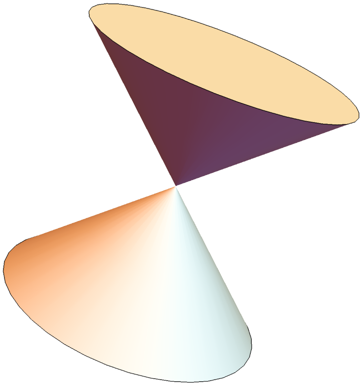

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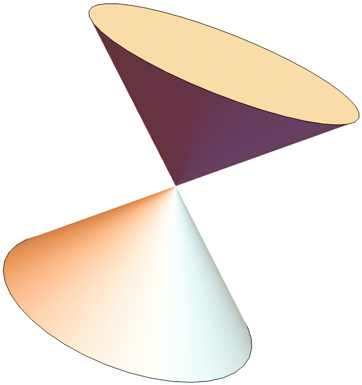

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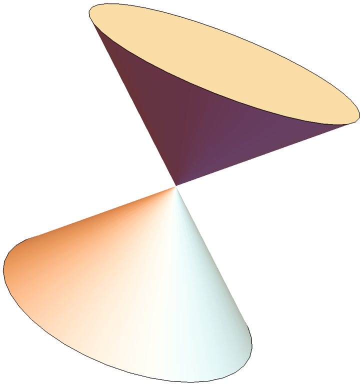

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$\Downarrow$

$\boxed{\text{time-dependent orbifold models}}$

Orbifolds

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- manifold M
- (Lie) group G
- *stabilizer* $G_p = \{g \in G \mid gp = p \in M\}$
- *orbit* $Gp = \{gp \in M \mid g \in G\}$
- charts $\phi = \pi \circ \mathscr{P}$ where:
  - $\mathscr{P}\colon U \subset \mathbb{R}^n \to U/G$
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- global orbit space M/G
- G group of isometries
- fixed points
- additional d.o.f. (*twisted states*)
- singular limits of CY manifolds

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time-dependent orbifolds

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Use **time-dependent orbifolds** to model **space-like singularities**:

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Divergences

Even in simple models (e.g. NBO, more on this later) the 4 tachyons amplitude is divergent **at tree level**:

$$A_4 \sim \int\limits_{q \sim \infty} \frac{\mathrm{d}q}{|q|} \mathscr{A}(q)$$

where

$$\mathscr{A}_{\text{closed}}(q) \sim q^{4-\alpha' \| \vec{p}_\perp \|^2} \qquad \text{and} \qquad \mathscr{A}_{\text{open}}(q) \sim q^{1-\alpha' \| \vec{p}_\perp \|^2} \operatorname{tr}([T_1,\ T_2]_+ [T_3,\ T_4]_+)$$

Null Boost Orbifold

CCC

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